Journal of Engineering Mathematics 17 (1983) 223–238

Observations on thermoconvection for bilayers in containers of arbitrary shape

R. NARAYANAN

Department of Chemical Engineering, University of Florida, Gainesville, FL 32611, U.S.A.

(Received November 23, 1982)

Summary

We have employed ideas on self-adjoint operators to formulate the problem on thermoconvection in bilayers and in bounded geometries. In particular, use of such operators helps in obtaining the behavior of the critical Rayleigh number with respect to domain size, fluid properties and boundary conditions. Two important cases are considered. These are (a) the case of a fluid bounded above by a solid of finite conductivity and height and (b) the case of two immiscible fluids which are vertically stacked against each other.

1. Introduction

Thermal convection in fluid layers is a well-known phenomenon. A number of very good review articles by Koschmieder [1], Rogers [2] and Busse [3] have explained various developments in the area. The superb two-volume treatise by Joseph [4] goes into considerable detail on various nuances of the thermal stability problem. This article is concerned with convection in bilayers. We are concerned, here, with two possible cases. These are (a) a fluid layer underneath a finite solid slab of finite conductivity but bounded below by a solid of infinite conductivity and (b) two fluid layers of finite depths but bounded by two horizontal plates of finite or infinite conductivity. The first case is sometimes known as the Rayleigh-Jeffreys problem. This problem has been considered earlier by Hurle, Jakeman and Pike [5], Nield [6] and Busse and Riahi [7]. These studies deal with the phenomenon for fluids of infinite lateral extent. The second case (also for infinite fluid layers) received attention by Smith [8], Zeren and Reynolds [9] and others. We shall deal with a simpler problem for case (b) in order to exemplify the method of analysis.

The purpose of this paper is to obtain some simple results from the theory of differential operators for both cases with the important difference that we are now concerned with fluids that are bounded laterally. We shall place only weak restrictions on the problem given by case (a) but the restrictions on case (b) will be stronger. Sections 2–4 will deal with case (a) and involve problem formulation, properties of the steady convective solutions and differential inequalities obtained from the linerarized stability problem of the quiescent solution. Sections 5–6 deal analogously with case (b). The details for derivations concerning case (b) will be limited if the anologous details have been given earlier for case (a). Many of the methods used in this paper are similar to those of

Narayanan [10] for the case of thermohaline convection. These methods involve the use of self-adjoint operators and the Fredholm alternative (Stakgold [11]).

2. Problem formulation for case (a)

We shall consider convection due to thermal gradients in a vertical container of arbitrary cross-section. An example is provided in Fig. 1 and the dimensionless depth of the fluid is unity. We invoke the Boussinesq approximation (Mihaljan [12]) and a Newtonion constitutive equation is assumed. In whatever that follows, bold-face variables are either vectors, tensors or matrix operators.

The governing equations in the fluid phase are:

$$\text{Continuity: } \nabla \cdot \boldsymbol{V} = 0, \tag{2.1}$$

$$\text{Motion: } \frac{1}{\text{Pr}}\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\nabla P + \frac{\text{Ra}}{\text{Pr}} T\boldsymbol{F} + \nabla^2 \boldsymbol{V}, \tag{2.2}$$

$$\text{Energy: } \frac{\partial T}{\partial t} + \text{Pr}(\boldsymbol{V} \cdot \nabla T) = \nabla^2 T. \tag{2.3}$$

The energy equation in the solid medium of finite conductivity is:

$$\alpha^* \frac{\partial T^s}{\partial t} = \nabla^2 T^s. \tag{2.4}$$

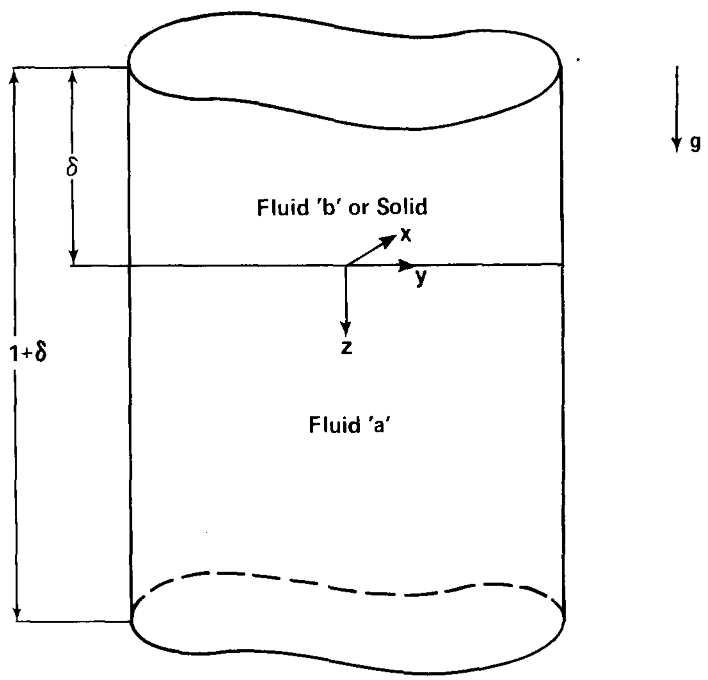

Fig. 1. Schematic of physical problem.

Here V and T are velocity and temperature fields. Pr and Ra are Prandtl and Rayleigh numbers, respectively, and $\boldsymbol{F}$ is a unit dimensionless force acting in the direction of gravity. Ra is defined as

$$\mathrm{Ra} \equiv \bar{\alpha} g (T_1 - T_0) L^3 / \alpha_T \nu, \tag{2.5}$$

where T_0 and T_1 represent the fluid temperatures at top and bottom of the fluid layer. α_T is the thermal diffusivity of the fluid, ν is the kinematic viscosity, L is the depth of the fluid and δ the thickness of solid, α^* is the ratio of thermal diffusivities of fluid and solid and $\bar{\alpha}$ is the thermal expansion coefficient. The following variables are used for the purpose of rendering the equations dimensionless:

$$V = V^* L / \nu, \tag{2.6a}$$

$$P = P^* L^2 / \rho_0 \nu^2, \tag{2.6b}$$

$$t = t^* k / \rho_0 C_p L^2, \tag{2.6c}$$

$$T = (T^* - T_1)/(T_0 - T_1). \tag{2.6d}$$

Asterisked variables are dimensioned. We shall restrict our study to situations where

$$\mathrm{Ra}(\boldsymbol{F} \times \nabla T_c) = \boldsymbol{0}. \tag{2.7}$$

This is a necessary condition for an initial temperature field T_c to exist in conjuction with a quiescent (or zero-velocity field) case. We now define the following matrix differential operator which will be used in the next section:

$$\boldsymbol{L} = \begin{bmatrix} \nabla^2 & 0 & 0 & 0 & 0 \\ 0 & \nabla^2 & 0 & 0 & 0 \\ 0 & 0 & \nabla^2 & \dfrac{\mathrm{Ra}}{\mathrm{Pr}} & 0 \\ 0 & 0 & \dfrac{\mathrm{Ra}}{\mathrm{Pr}} & \dfrac{\mathrm{Ra}}{\mathrm{Pr}^2}\nabla^2 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\beta\,\mathrm{Ra}}{\mathrm{Pr}^2}\nabla^2 \end{bmatrix}, \tag{2.8}$$

where β will be defined later; ∇^2 is the usual Laplacian operator.

We may define a five-vector $\boldsymbol{Q}$ such that

$$\boldsymbol{Q}^t \equiv [U, V, W, T, T^s], \tag{2.9}$$

where U, V, W are the rectangular cartesian components of V and 't' represents the transpose operation.

Following Malkus and Veronis [13] and Schluter et al. [14] we introduce the following perturbation series for the steady fields of $\boldsymbol{Q}$ and P:

$$\boldsymbol{Q} = \boldsymbol{Q}_c + \varepsilon \boldsymbol{Q}_0 + \varepsilon^2 \boldsymbol{Q}_1 + O(\varepsilon^3), \tag{2.10}$$

$$P = P_c + \varepsilon P_0 + \varepsilon^2 P_1 + O(\varepsilon^3), \tag{2.11}$$

$$\mathrm{Ra} = \mathrm{Ra}_0 + \varepsilon\, \mathrm{Ra}_1 + O(\varepsilon^2). \tag{2.12}$$

Here ε is an indicator of the deviation from the quiescent solution; $\boldsymbol{Q}_c$ and P_c are the solution fields in the quiescent state. The perturbed velocity fields will be denoted by $\boldsymbol{U}_i$ and the perturbed temperature fields will be denoted by $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_i^s$. We may substitute the above perturbation series into the governing equations (2.1)–(2.4) and obtain for terms of $O(\varepsilon)$:

$$\boldsymbol{L}\boldsymbol{Q}_0 = \boldsymbol{d}P_0 \tag{2.13}$$

where

$$\boldsymbol{d}^t = (\partial/\partial x, \partial/\partial y, \partial/\partial z, 0, 0). \tag{2.14}$$

Clearly the $\boldsymbol{U}_i$ are solenoidal. The casting of the thermoconvective equations into matrix operator form is not new (Joseph [4], Narayanan [10]).

The boundary conditions for the fluid are

$$\boldsymbol{U}_i \cdot \boldsymbol{n} = 0 \qquad \text{along } \partial S, \tag{2.15}$$

$$\boldsymbol{S}_i \cdot \boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{S}_i \cdot \boldsymbol{n}) = \boldsymbol{0} \qquad \text{on } \partial S_F, \tag{2.16}$$

$$\boldsymbol{U}_i = 0 \quad \text{on } \partial S \cap \partial S_F, \tag{2.17}$$

where $\boldsymbol{S}_i$ is the extra stress tensor. We also have

$$\nabla \theta_i \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial S_N, \tag{2.18}$$

$$\theta_i = 0 \qquad \text{on } \partial S_D, \tag{2.19}$$

where ∂S_D includes the bottom surface.

The boundary conditions for the solid medium are

$$\theta_i^s = 0 \qquad \text{on } \partial S_D^s \tag{2.20}$$

where ∂S_D^s includes the top surface of the solid and

$$\nabla \theta_i^s \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial S_N^s. \tag{2.21}$$

The sides of the container will support Dirichlet or Neumann conditions. Along the common boundary between the fluid and solid, we have

$$\nabla \theta_i \cdot \boldsymbol{n} = \beta \nabla \theta_i^s \cdot \boldsymbol{n} \tag{2.22}$$

and

$$\theta_i = \theta_i^s, \tag{2.23}$$

where β represents the ratio of thermal conductivities. A Fourier law has implicitly been assumed. $\boldsymbol{n}$ is the unit outward normal and ∂S as well as ∂S^s are the closed bounding surfaces of $\hat{V}$ and $\hat{V}_S$—the fluid and solid domains, respectively. It is also noteworthy that the equations in the solid and fluid media are uncoupled from each other only if a Nusselt number is introduced. In such an event equations (2.22) and (2.23) would be replaced by

$$\nabla\theta_i \cdot \boldsymbol{n} + \mathrm{Nu}\,\theta_i = 0 \tag{2.24}$$

where $\mathrm{Nu}(\boldsymbol{x})$ is a positive real-valued function of position. We finally note that the boundary data is often abreviated by $B(\boldsymbol{Q}_i) = 0$.

3. Properties of convective motion for case (a)

We may define an inner product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ such that

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \int_V \boldsymbol{a} \cdot \boldsymbol{b}\,\mathrm{d}V \tag{3.1}$$

where V is the domain of the integral operator in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined.

We can easily see that $\boldsymbol{L}$ is self-adjoint when the above inner product is utilized. We shall first state a few properties which can easily be proved.

Property 3.1. The bifurcation point Ra_0 cannot be negative.

Property 3.2. Steady convective motion cannot exist if $\mathrm{Ra} < 0$. We also have $\mathrm{Ra}_1 = 0$, i.e. there exists no subcritical steady motion.

The proofs follow standard methods (Joseph [4]) and utilize the fact that $\boldsymbol{L}$ is self-adjoint and $\boldsymbol{U}_0$ is solenoidal. Property 3.1 follows when we take the inner product of equation (2.13) with $\boldsymbol{Q}_0^*$ (the complex conjugate of $\boldsymbol{Q}_0$). Property 3.2 follows on considering terms of $O(\varepsilon^2)$. We have

$$\boldsymbol{L}\boldsymbol{Q}_1 = \boldsymbol{d}P_1 + \boldsymbol{h}_1 \tag{3.2}$$

where

$$\boldsymbol{h}_1^t \equiv \left[(\boldsymbol{U}_0 \cdot \nabla \boldsymbol{U}_0)_x, (\boldsymbol{U}_0 \cdot \nabla \boldsymbol{U}_0)_y, (\boldsymbol{U}_0 \cdot \nabla \boldsymbol{U}_0)_z - \frac{\mathrm{Ra}_1}{\mathrm{Pr}}\theta_0, \frac{\mathrm{Ra}_0}{\mathrm{Pr}}\boldsymbol{U}_0 \cdot \nabla\theta_0, 0\right]. \tag{3.3}$$

On taking the inner products of equation (2.13) with $\boldsymbol{Q}_1$ and equation (3.2) with $\boldsymbol{Q}_0$ and evaluating the difference we can see that $\mathrm{Ra}_1 = 0$. The solenoidal character of $\boldsymbol{U}_0$, $\boldsymbol{U}_1$ and self-adjointness of $\boldsymbol{L}$ are instrumental in the proof. From here on we shall always assume that $\mathrm{Ra}_0 > 0$.

We can consider the dependence of Ra_0 on β by taking the derivatives of equation (2.13) and the boundary conditions with respect to β. Tilde ($\tilde{}$) overbars represent differentiation with respect to β. We obtain

$$\boldsymbol{L}\tilde{\boldsymbol{Q}}_0 = \boldsymbol{d}\tilde{P}_0 + \tilde{\boldsymbol{h}}_0 \tag{3.4}$$

where

$$\tilde{\boldsymbol{h}}_0' = \left[0, 0, \frac{-\theta_0}{\mathrm{Pr}}\widetilde{\mathrm{Ra}}_0, 0, 0\right]. \tag{3.5}$$

The boundary conditions retain a similar form except for equation (2.22) which is replaced by:

$$\nabla\theta_0 \cdot \boldsymbol{n} = \beta\nabla\tilde{\theta}_0^s \cdot \boldsymbol{n} + \nabla\theta_0^s \cdot \boldsymbol{n}. \tag{3.6}$$

We can take the inner product of equation (3.4) with $\boldsymbol{Q}_0$ and likewise the inner product of equation (2.13) with $\tilde{\boldsymbol{Q}}_0$. On evaluating the difference and using the fact that $\tilde{\boldsymbol{U}}_0$ is solenoidal we get

$$\langle \boldsymbol{Q}_0, \tilde{\boldsymbol{h}}_0 \rangle = \int_{\hat{V}} \frac{\mathrm{Ra}_0}{\mathrm{Pr}^2}\left(\theta_0\nabla^2\tilde{\theta}_0 - \tilde{\theta}_0\nabla^2\theta_0\right)\mathrm{d}\hat{V}$$
$$+ \int_{\hat{V}_s} \frac{\beta\,\mathrm{Ra}_0}{\mathrm{Pr}^2}\left(\theta_0^s\nabla^2\tilde{\theta}_0^s - \tilde{\theta}_0^s\nabla^2\theta_0^s\right)\mathrm{d}\hat{V}_s. \tag{3.7}$$

Equations (3.5), (3.6) and (3.7) give $\partial\,\mathrm{Ra}_0/\partial\beta > 0$. Thus we have the following property:

Property 3.3. For thermoconvective motion in a bounded container, the critical Ra increases monotonically with β.

Property 3.3 agrees with the trend shown by the numerical results of Nield [6]. We now consider the change of Ra_0 with domain size. The effect of δ is seen by differentiating equation (2.13). The 'z' co-ordinate in $\hat{V}_s$ is immobolized by using a transformation $\tilde{z} = z/\delta$. If '$\hat{}$' overbars represent derivatives with respect to δ, then we see that

$$\boldsymbol{L}\hat{\boldsymbol{Q}}_0 = \boldsymbol{d}\hat{P}_0 + \hat{\boldsymbol{h}}_0 \tag{3.8}$$

Here

$$\hat{\boldsymbol{h}}_0' = \left[0, 0 - \frac{\hat{\mathrm{Ra}}_0\theta_0}{\mathrm{Pr}}, 0, \frac{\mathrm{Ra}_0 2\beta}{\delta\,\mathrm{Pr}^2}\frac{\partial^2\theta_0^s}{\partial z^2}\right]. \tag{3.9}$$

On taking the inner products of equation (3.8) with $\boldsymbol{Q}_0$ and equation (2.13) with $\hat{\boldsymbol{Q}}_0$ and then evaluating the difference we obtain:

$$\frac{-\hat{\mathrm{Ra}}_0}{\mathrm{Pr}}\int_{\hat{V}}\theta_0 W_0\,\mathrm{d}\hat{V} + \frac{2\beta\,\mathrm{Ra}_0}{\delta\,\mathrm{Pr}^2}\int_{\hat{V}_s}\theta_0^s\frac{\partial^2\theta_0^s}{\partial z^2}\mathrm{d}\hat{V}_s = 0. \tag{3.10}$$

Equation (3.10) does not yield any definitive result since the behavior of the second integral is unknown. We shall obtain a stronger result later.

The behavior of Ra_0 with lateral domain size is inspected by considering the restricted case of a rectangular box whose sides extend to L_1 in the 'x' direction and L_2 in the 'y' direction. We may immobolize the 'x' co-ordinate in both (fluid and solid) domains. This is done by introduction of a new variable $\tilde{x} = x/L_1$. We differentiate equation (2.13) with respect to L_1 and obtain

$$L\bar{Q}_0 = d\bar{P}_0 + \bar{h}_1 + \frac{2}{L_1} dP_0 + \bar{h}_2. \tag{3.11}$$

The overbars (-) represent derivatives with respect to L_1. Also

$$\bar{h}_1^t = \frac{-2}{L_1}\left[\nabla_H^2 U_0,\ \nabla_H^2 V_0,\ \nabla_H^2 W_0,\ \frac{\mathrm{Ra}_0}{\mathrm{Pr}^2}\nabla_H^2\theta_0,\ \frac{\beta \mathrm{Ra}_0}{\mathrm{Pr}^2}\nabla_H^2\theta_0^s\right] \tag{3.12}$$

and

$$\bar{h}_2^t = \left[\frac{-1}{L_1}\frac{\partial P_0}{\partial x},\ 0,\ \frac{-\theta_0}{\mathrm{Pr}}\overline{\mathrm{Ra}}_0 - \frac{2}{L_1}\frac{\mathrm{Ra}_0}{\mathrm{Pr}}\theta_0,\ \frac{-2}{L_1}\frac{\mathrm{Ra}_0}{\mathrm{Pr}}W_0,\ 0\right]. \tag{3.13}$$

∇_H^2 represents the two-dimensional Laplacian in 'y' and 'z'. We also observe that

$$\nabla \cdot \bar{U}_0 = \frac{1}{L_1}\frac{\partial U_0}{\partial x}. \tag{3.14}$$

On taking the inner product of equation (3.11) with Q_0 and equation (2.13) with $\bar{Q}_0$, we get

$$\langle Q_0, L\bar{Q}_0\rangle, -\langle \bar{Q}_0, LQ_0\rangle = \langle Q_0, d\bar{P}_0\rangle - \langle \bar{Q}_0, dP_0\rangle + \langle Q_0, (\bar{h}_1 + \bar{h}_2)\rangle. \tag{3.15}$$

The left-hand side of equation (3.15) is zero because L is self-adjoint. The right-hand side in conjunction with equation (3.14) and a solenoidal U_0 yields

$$\int_{\hat{V}} -\theta_0\nabla^2\theta_0\left(\overline{\mathrm{Ra}}_0 + \frac{4}{L_1}\mathrm{Ra}_0\right)d\hat{V} > 0. \tag{3.16}$$

This leads us to Property (3.4).

Property 3.4. The bifurcation points Ra_0 of the Boussinesq equations in a rectangular box behave such that

$$\frac{\partial}{\partial L_1}(\mathrm{Ra}_0 L_1^4) > 0.$$

Similar behavior has been shown for axisymmetric motion in a right circular cylinder by Vrentas et al. [16].

Linear stability of quiescent solution for case (a)

We may consider the stability of the conductive state by perturbing Q_c with $Q'\,e^{st}$. Here s is complex and Q' is infinitesimal so that introduction of the perturbation into the

Boussinesq equations (2.1)–(2.4) and attendant boundary data yields

$$LQ' = dP' + sQ'_t, \qquad B(Q') = 0. \tag{4.1}$$

Here

$$Q'_t \equiv \left(\frac{U'}{\mathrm{Pr}}, \frac{V'}{\mathrm{Pr}}, \frac{\mathrm{Ra}\, W'}{\mathrm{Pr}}, \frac{\mathrm{Ra}\, \theta'}{\mathrm{Pr}^2}, \frac{\alpha^* \,\mathrm{Ra}\, \beta}{\mathrm{Pr}^2} \theta'^s \right).$$

Property 4.1. s is real. The proof is quite standard (Sherman and Ostrach [15]) and requires premultiplication of (4.1) and its conjugate by Q'^* and Q', respectively. Self-adjointness of L and the solenoidal character of U' and U'^* assures that $s = s^* = \sigma$ (say). We now consider necessary conditions for $(\partial\sigma/\partial\,\mathrm{Ra})_{\mathrm{Ra}_0} > 0$. Thus we differentiate equation (4.1) with respect to Ra and remember that $\sigma = 0$ when $\mathrm{Ra} = \mathrm{Ra}_0$. We obtain

$$L\hat{Q}' = d\hat{P}' + Q'_t \hat{\sigma} + \hat{h}. \tag{4.2}$$

Here $\hat{h}$ is a five-vector, and is null except for the third element, which is equal to $-\theta'/\mathrm{Pr}$. The '$\hat{\ }$' overbars represent derivatives with respect to Ra.

Premultiplication of equation (4.2) by Q' and equation (4.1) by $\hat{Q}'$, setting $\sigma = 0$ in equation (4.1) and finally evaluating the difference of inner products yields

$$(\hat{\sigma})_{\mathrm{Ra}_0} \left\{ \int_V \left(\frac{|U'|^2}{\mathrm{Pr}} + \theta'^2 \frac{\mathrm{Ra}_0}{\mathrm{Pr}^2} \right) \mathrm{d}\hat{V} + \int_{V_s} (\theta'^s)^2 \frac{\alpha^* \,\mathrm{Ra}_0 \beta}{\mathrm{Pr}^2} \mathrm{d}\hat{V}_s \right\} = \int_{\hat{V}} \frac{W'\theta'}{\mathrm{Pr}} \mathrm{d}V. \tag{4.3}$$

This leads to Property 4.2.

Property 4.2. If $(\hat{\sigma})_{\mathrm{Ra}_0} > 0$ then it necessarily follows that

$$\left\{ \int_{\hat{V}} \frac{|U'|^2}{\mathrm{Pr}} + \theta'^2 \frac{\mathrm{Ra}_0}{\mathrm{Pr}^2} \mathrm{d}\hat{V} + \int_{\hat{V}_s} (\theta'^s)^2 \alpha^* \frac{\mathrm{Ra}_0 \beta}{\mathrm{Pr}^2} \mathrm{d}\hat{V}_s \right\} > 0. \tag{4.4}$$

The left-hand side of the above inequality will be abreviated by E.

Differentiation of equation (4.1) and the boundary data with respect to β and employment of earlier methodology gives

$$\left(\frac{\partial\sigma}{\partial\beta} \right)_{\mathrm{Ra}_0} \frac{E\,\mathrm{Pr}^2}{\mathrm{Ra}_0} = \int_{\partial S^s} \left[\theta'^s \nabla\theta'^s \cdot \boldsymbol{n}|_\xi \right] \mathrm{d}S^s \tag{4.5}$$

where ξ is the location of the common boundary ∂S^s between fluid and solid. The term in the square bracket of the right-hand side of equation (4.5) can symbollically be replaced by $-Nu(\theta')^2/\beta$ which must be negative.

We therefore have the following result

Property 4.3. $(\partial\sigma/\partial\beta)_{\mathrm{Ra}_0} < 0$ provided $(\partial\sigma/\partial\,\mathrm{Ra})_{\mathrm{Ra}_0} > 0$.

We finally end this section by considering the effect of solid thickness δ on σ. Using a proof similar to that for obtaining equation (3.10) we get

$$\left(\frac{\partial\sigma}{\partial\delta}\right)_{Ra_0} E = \frac{-2}{\delta}\frac{Ra_0\beta}{Pr^2}\int_{\hat{V}_s}\theta'^s\frac{\partial^2\theta'^s}{\partial z^2}d\hat{V}_s. \tag{4.6}$$

We now assert that the eigenvalues of equation (2.13) i.e. Ra_0 are simple so that

$$Q' = \gamma Q_0 \tag{4.7}$$

where γ is a nonzero finite constant. We obtain

$$\left(\frac{\partial\sigma}{\partial\delta}\right)_{Ra_0} \tilde{E}\gamma^2 = \frac{-2\beta}{\delta}\frac{Ra_0}{Pr^2}\gamma^2\int_{\hat{V}_s}\theta_0^s\frac{\partial^2\theta_0^s}{\partial z^2}d\hat{V}_s \tag{4.8}$$

where

$$\tilde{E} = E/\gamma^2. \tag{4.9}$$

Comparison of equations (4.8) and (3.10) yields

$$\left(\frac{\partial\sigma}{\partial\delta}\right)_{Ra_0} \Big/ \frac{\partial\, Ra_0}{\partial\delta} < 0. \tag{4.10}$$

Thus we get a condition which is stronger than equation (3.10), i.e. Property 4.4.

Property 4.4. If the eigenvalues Ra_0 are simple and if $(\partial\sigma/\partial Ra)_{Ra_0} > 0$ then inequality (4.10) is true.

5. Problem formulation for case (b)

The governing equations for convection in bounded bilayers of fluid are given below. For fluid 'a' we have
Equations of motion:

$$\frac{1}{Pr_a}\frac{\partial V^a}{\partial t} + V^a\cdot\nabla V^a = -\nabla P^a + \frac{Ra^a}{Pr_a}T^a F + \nabla^2 V^a. \tag{5.1}$$

Equation of energy:

$$\frac{\partial T^a}{\partial t} + Pr_a V^a\cdot\nabla T^a = \nabla^2 T^a. \tag{5.2}$$

For fluid 'b' we have
Equations of motion:

$$\frac{\alpha^*}{Pr_b}\frac{\partial V^b}{\partial t} + V^b\cdot\nabla V^b = -\nabla P^b + \frac{Ra^b}{Pr_b\delta^4}T^b F + \nabla^2 V^b. \tag{5.3}$$

Equation of energy:

$$\alpha^* \frac{\partial T^b}{\partial t} + \mathrm{Pr}_b V^b \cdot \nabla T^b = \nabla^2 T^b. \tag{5.4}$$

V^a and V^b are solenoidal velocity fields. Here Ra^a and Ra^b are the Rayleigh numbers and are defined as usual. The expression for Ra^a contains the temperature difference between the horizontal planes that bound fluid 'a'. The expression Ra^b is defined in an analogous fashion. α^* represents the ratio of thermal diffusivities in each phase.

We define a few terms which will arise after introduction of the usual perturbation series:

$$L_1 = \begin{bmatrix} \nabla^2/K_1 & & & & \\ & \nabla^2/K_1 & & & \\ & & \nabla^2/K_1 & \delta\,\mathrm{Pr}_a & \\ & & \delta\,\mathrm{Pr}_a & \delta\nabla^2 & \\ & & & & \{0\} \end{bmatrix} \tag{5.5}$$

All the elements without entries take on zero values. {0} represents a null matrix of fourth order and L_1 is therefore an eighth-order matrix with only six nonzero elements. Likewise:

$$L_2 = \begin{bmatrix} \{0\} & & & & \\ & \nabla^2/K_2 & & & \\ & & \nabla^2/K_2 & & \\ & & & \nabla^2/K_2 & \mathrm{Pr}_b\alpha\beta \\ & & & \mathrm{Pr}_b\alpha\beta & \delta\alpha\beta\nabla^2 \end{bmatrix}, \tag{5.6}$$

$$K_1 = \frac{\mathrm{Ra}^a}{\mathrm{Pr}_a^2\delta} \tag{5.7}$$

and

$$K_2 = \frac{\mathrm{Ra}^b}{\alpha\beta\delta^4\,\mathrm{Pr}_b^2}. \tag{5.8}$$

We introduce the following eight-vector

$$Q^t = [U^a, V^a, W^a, T^a, U^b, V^b, W^b, T^b] \tag{5.9}$$

and as before U, V, W are the rectangular cartesian components of V. We will now consider the steady fields for Q, P^a and P^b, As in the analysis of case (a) we shall expand Q, P^a, and P^b in a perturbation series with ε as the series parameters.

$$Q = Q_c + \varepsilon Q_0 + O(\varepsilon^2), \tag{5.10}$$

$$P^a = P_c^a + \varepsilon P_0^a + O(\varepsilon^2), \tag{5.11}$$

$$P^b = P_c^b + \varepsilon P_0^b + O(\varepsilon^2), \tag{5.12}$$

$$\mathrm{Ra}^a = \mathrm{Ra}_0^a + \varepsilon\, \mathrm{Ra}_1^a + O(\varepsilon^2) \qquad \text{for fixed } \mathrm{Ra}^b, \tag{5.13a}$$

$$\mathrm{Ra}^b = \mathrm{Ra}_0^b + \varepsilon\, \mathrm{Ra}_1^b + O(\varepsilon^2) \qquad \text{for fixed } \mathrm{Ra}^a. \tag{5.13b}$$

Once again the subscript 'c' reflects the linear or conduction profile which is compatible to a no flow situation. On substituting the above series into the governing equations we obtain

$$(\boldsymbol{L}_1 + \boldsymbol{L}_2)\boldsymbol{Q}_0 = \frac{1}{K_1}\boldsymbol{d}^a P_0^a + \frac{1}{K_2}\boldsymbol{d}^b P_0^b, \tag{5.14}$$

where $\boldsymbol{d}^a$ and $\boldsymbol{d}^b$ are given by the following eight vectors

$$\boldsymbol{d}^a \equiv (\partial/\partial x, \partial/\partial y, \partial/\partial z, 0, 0, 0, 0, 0), \tag{5.15a}$$

$$\boldsymbol{d}^b \equiv (0, 0, 0, 0, \partial/\partial x, \partial/\partial y, \partial/\partial z, 0). \tag{5.15b}$$

The perturbed velocity and temperature fields are denoted by $\boldsymbol{U}_i^a$, θ_i^a and $\boldsymbol{U}_i^b$, θ_i^b, respectively. We can also see that $\boldsymbol{U}_i^a$ and $\boldsymbol{U}_i^b$ are solenoidal.

The boundary conditions for each fluid phase are as follows:

$$\boldsymbol{U}_i^{a,b} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial S^{a,b}, \tag{5.16}$$

$$\boldsymbol{S}_i^{a,b} \cdot \boldsymbol{n} - \boldsymbol{n}\left(\boldsymbol{n} \cdot \boldsymbol{S}_i^{a,b} \cdot \boldsymbol{n}\right) = \boldsymbol{0} \qquad \text{on } \partial S_F^{a,b}, \tag{5.17}$$

$$\boldsymbol{U}_i^{a,b} = \boldsymbol{0} \qquad \text{on } \partial S^{a,b} \cap \partial S_F^{a,b}. \tag{5.18}$$

where $\boldsymbol{S}_i$ is the extra stress tensor, ∂S represents the bounding surface for each phase and ∂S_F is a surface that includes the common area between both phases. The above conditions are clearly somewhat restricted since they assume the case of no surface deflection and a traction-free common interphase. The assumption of a flat interphase is reasonable provided the fluid layers are not thin. It appears that the deflection of the free surface plays only a small role in the experiments of Hoard et al. [17] and Palmer and Berg [18]. A traction-free interphase is a strong restriction which is imposed in order to contain ourselves to a simpler problem. It is noteworthy that there is another physically realizable case, that falls within the category of the boundary conditions, which are considered above. This is the situation of two fluids separated by a thin, highly conductive rigid membrane. We also note that the effects of surface tension gradients are not considered here.

The boundary conditions on the perturbed temperature field are

$$\nabla\theta_i^{a,b} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial S_N^{a,b}, \tag{5.19}$$

$$\theta_i^{a,b} = 0 \qquad \text{on } \partial S_D^{a,b}, \tag{5.20}$$

$$\nabla\theta_i^{a,b} \cdot \boldsymbol{n} + \mathrm{Nu}^{a,b}\theta_i^{a,b} = 0 \qquad \text{on } \partial S_R^{a,b} \tag{5.21}$$

where $Nu^{a,b}$ is a real-valued positive function of position. Also

$$\partial S_R^{a,b} = \partial S_W^{a,b} \cap \left(\partial S_N^{a,b} \cup \partial S_D^{a,b}\right) \tag{5.22}$$

where $\partial S_W^{a,b}$ represents the bounding surface excluding the common boundary between the two fluid phases. The side walls support Dirichlet or Neumann conditions and the horizontal surfaces that bound the container support Robin conditions in general. Along the common boundary we have

$$\theta_i^a = \alpha\theta_i^b, \tag{5.23}$$

$$\nabla\theta_i^a \cdot \boldsymbol{n} = \beta\nabla\theta_i^b \cdot \boldsymbol{n}, \tag{5.24}$$

with

$$\alpha = (T_{-\delta} - T_s)/(T_s - T_1), \tag{5.25}$$

where $T_{-\delta}$ represents the temperature of the top surface, T_s the temperature of the common surface and T_1 is the temperature of the bottom surface and

$$\beta = \alpha k^b/k^a \tag{5.26}$$

i.e. β is related to the ratio of thermal conductivities. A weak restriction that we shall place is

$$\int_{\partial S^a} \theta_i^a \nabla\theta_i^a \cdot \boldsymbol{n}\,\mathrm{d}S^a \quad \text{and} \quad \int_{\partial S^b} \theta_i^b \nabla\theta_i^b \cdot \boldsymbol{n}\,\mathrm{d}S^b \leqslant 0. \tag{5.27}$$

The boundary data is abbreviated as $B(\boldsymbol{Q}_i) = 0$. It turns out that this is true automatically when $\mathrm{Nu}^{a,b} \to \infty$. It is noteworthy that θ_i^a and $\nabla\theta_i^a \cdot \boldsymbol{n}$ at the common boundary are related by a Robin condition of the form of equation (2.24) but this serves only to decouple the events in both places.

Using the same methodology as in Section 4 we have

$$(\boldsymbol{L}_1 + \boldsymbol{L}_2)\boldsymbol{Q}' = \frac{\boldsymbol{d}^a P'^a}{K_1} + \frac{\boldsymbol{d}^b P'^b}{K_2} + s\boldsymbol{Q}_t' \tag{5.28}$$

and

$$B(\boldsymbol{Q}') = 0 \tag{5.29}$$

where $\boldsymbol{Q}_t'$ is an eight-vector given by

$$\boldsymbol{Q}_t' \equiv \left(\frac{U'^a}{\mathrm{Pr}_a K_1}, \frac{V'^a}{\mathrm{Pr}_a K_1}, \frac{W'^a}{\mathrm{Pr}_a K_1}, \delta\theta'^a, \frac{\alpha^* U'^b}{\mathrm{Pr}_b K_2}, \frac{\alpha^* V'^b}{\mathrm{Pr}_b K_2}, \frac{\alpha^* W'^b}{\mathrm{Pr}_b K_2}, \delta\beta\alpha\alpha^*\theta'^b\right). \tag{5.30}$$

6. Inequalities for steady convection and linearized stability for case (b)

The methods in this section follow closely those of Sections 3 and 4. We will delete many of the algebraic details. They can be obtained from the author on request. We will be concerned with the problem for fixed Ra^b, therefore equation (5.13a) will be used.

Property 6.1. For the thermoconvective problem given by case (b) we have

$$\partial\, \mathrm{Ra}_0^a/\partial\, \mathrm{Ra}^b < 0.$$

We prove this by differentiating equation (5.14) and the boundary data with respect to Ra_b. We observe that $(\boldsymbol{L}_1 + \boldsymbol{L}_2)$ is indeed self-adjoint with respect to the inner product given by equation (3.1). Further, $B(\boldsymbol{Q}_0)$ is also self-adjoint.

On application of the Fredholm alternative or in other words the methodology in proofs of properties in Sections 3 and 4 we have

$$\frac{\hat{\mathrm{Ra}}_0^a}{K_1 \mathrm{Pr}_a^2}\int_{\hat{V}_a} \theta_0^a \nabla^2 \theta_0^a \mathrm{d}\hat{V}_a = -\int_{\hat{V}_b} \theta_0^b \nabla^2 \theta_0^b \frac{\delta}{\mathrm{Pr}_b^2 K_2} \mathrm{d}\hat{V}_b \tag{6.1}$$

where $\hat{V}_a$ and $\hat{V}_b$ are taken over the domains of fluid 'a' and fluid 'b' and $\hat{\mathrm{Ra}}_0$ denotes $\partial\, \mathrm{Ra}_0^a/\partial\, \mathrm{Ra}^b$. Equation (6.1) yields Property 6.1.

We consider the variation with respect to β by differentiating equation (5.14) and the attendant boundary conditions with respect to β. We obtain

$$(\boldsymbol{L}_1 + \boldsymbol{L}_2)\tilde{\boldsymbol{Q}} = \frac{1}{K_1}\boldsymbol{d}^a \tilde{P}_0^a + \frac{1}{K_2}\boldsymbol{d}^b \tilde{P}_0^b + \tilde{\boldsymbol{h}}_0. \tag{6.2}$$

The tilde sign ($\tilde{}$) refers to derivatives with respect to β, and

$$\tilde{\boldsymbol{h}}_0' = \left(0, 0, -\frac{\widetilde{\mathrm{Ra}}_0^a \theta_0^a}{\mathrm{Pr}_a K_1}, 0, 0, 0, 0, 0\right). \tag{6.3}$$

The boundary conditions remain unchanged in form except at the interface where

$$\nabla\tilde{\theta}_0^a \cdot \boldsymbol{n} = \beta \nabla \tilde{\theta}_0^b \cdot \boldsymbol{n} + \nabla \theta_0^b \cdot \boldsymbol{n}. \tag{6.4}$$

Application of the Fredholm alternative yields

$$-\alpha\delta \int_{\partial S_c} \theta_0^b \left.\frac{\partial \theta_0^b}{\partial z}\right|_{z=\xi} \mathrm{d}S_c = \frac{-\widetilde{\mathrm{Ra}}_0^a}{\mathrm{Pr}_a K_1}\int_{\hat{V}_a} W_0^a \theta_0^a \mathrm{d}\hat{V}_a. \tag{6.5}$$

ξ is the location of the interface ∂S_c. The left-hand side of equation (6.5) is negative. We hence have Property 6.2.

Property 6.2. For the thermoconvective problem given by case (b) $\partial \mathrm{Ra}_0^a/\partial\beta > 0$. This result causes us to consider the behavior of Ra_0^a with Nu^b. Here Nu^b is a positive real-valued function and defined at the upper horizontal boundary of phase 'b'. Differentiating the equations given by (5.14) yields

$$(\boldsymbol{L}_1 + \boldsymbol{L}_2)\hat{\boldsymbol{Q}}_0 = \frac{\boldsymbol{d}^a \hat{\boldsymbol{P}}_0^a}{K_1} + \frac{\boldsymbol{d}^b \hat{\boldsymbol{P}}_0^b}{K_2} + \hat{\boldsymbol{h}}_0, \tag{6.6}$$

where '$\hat{}$' overbars reflect differentiation with respect to Nu^b. $\hat{\boldsymbol{h}}_0^t$ is identical to $\tilde{\boldsymbol{h}}_0^t$, given by equation (6.3), with the exception that '$\tilde{}$' signs are replaced by '$\hat{}$' overbars. The boundary conditions remain unchanged in form everywhere except at the top surface of fluid 'b'. The condition here is

$$\nabla\hat{\theta}_0^b \cdot \boldsymbol{n} + \mathrm{Nu}^b\hat{\theta}_0^b + \theta_0^b = 0. \tag{6.7}$$

The Fredholm alternative gives

$$\frac{\widehat{\mathrm{Ra}}_0^a}{\mathrm{Pr}_a K_1}\int_{\hat{V}_a}\theta_0^a W_0^a \mathrm{d}\hat{V}_a = \delta\alpha\beta\int_{\partial S_t}\left(\theta_0^b\right)^2\big|_{z=\xi}\mathrm{d}S_t \tag{6.8}$$

where ξ is the location of the top surface ∂S_t. This leads to *Property 6.3* which states that $\partial\mathrm{Ra}_0^a/\partial\mathrm{Nu}^b > 0$. An analogous result for thermal convection in single fluid layers has been shown by Joseph [4].

The domain size has an effect on Ra_0^a but the results can be made more definitive by considering the linearized stability problem. We state two properties.

Property 6.4. If the eigenvalues Ra_0^a for equation (5.14) are simple and

$$\left(\frac{\partial\sigma}{\partial\,\mathrm{Ra}^a}\right)_{\mathrm{Ra}_0^a} > 0,$$

then

$$\frac{(\partial\sigma/\partial\delta)_{\mathrm{Ra}_0^a}}{(\partial\,\mathrm{Ra}_0^a/\partial\delta)} < 0. \tag{6.9}$$

Property 6.5. For conditions of Property 6.4,

$$\frac{(\partial\sigma/\partial L_1)_{\mathrm{Ra}_0^a}}{(\partial\,\mathrm{Ra}_0^a/\partial L_1)} < 0. \tag{6.10}$$

The proofs for Properties (6.4) and (6.5) follow in the same manner as those of Property (4.4). We do not give the details of algebra here but indicate that a lemma, quite similar to Property (4.2) is needed. L_1 is the width of a rectangular container.

7. Summary and conclusions

We have employed parametric differentiation of self-adjoint operators and a modified form of the Fredholm alternative in order to arrive at a number of differential inequalities

Table 1
Summary of results

Parameter	Case (a)	Case (b)
β	$\frac{\partial \mathrm{Ra}_0}{\partial \beta} > 0, \left(\frac{\partial \sigma}{\partial \beta}\right)_{\mathrm{Ra}_0} < 0$	$\frac{\partial \mathrm{Ra}_0^a}{\partial \beta} > 0$
δ	$\left(\frac{\partial \sigma}{\partial \delta}\right)_{\mathrm{Ra}_0} / \frac{\partial \mathrm{Ra}_0}{\partial \delta} < 0$	$\left(\frac{\partial \sigma}{\partial \delta}\right)_{\mathrm{Ra}_0^a} / \frac{\partial \mathrm{Ra}_0^a}{\partial \delta} < 0$
$^*L_1$	$\frac{\partial}{\partial L_1}(\mathrm{Ra}_0 L_1^4) > 0$	$\left(\frac{\partial \sigma}{\partial L_1}\right)_{\mathrm{Ra}_0^a} / \frac{\partial \mathrm{Ra}_0^a}{\partial L_1} < 0$
Ra^b	–	$\frac{\partial \mathrm{Ra}_0^a}{\partial \mathrm{Ra}^b} < 0$ and
Nu^b	–	$\frac{\partial \mathrm{Ra}_0^a}{\partial \mathrm{Nu}^b} > 0$

A restriction is that $(\partial \sigma / \partial \mathrm{Ra})_{\mathrm{Ra}_0} > 0$ for case (a) and $(\partial \sigma / \partial \mathrm{Ra}^a)_{\mathrm{Ra}_0^a} > 0$ for case (b).
* For this parameter we consider only rectangular containers.

for thermoconvective motion in bilayers. These have been done for containers of arbitrary cross section and fairly general boundary conditions. We refer to Table 1.

For the case of the Rayleigh-Jeffreys problem our results on variation with the ratio of thermal conductivities agree with the trends shown by Nield [6]. The horizontal domain dependence yields definitive results on the variation of critical Rayleigh numbers. These results are similar to those of Vrentas et al. [16], who discuss the axisymmetric motion in a circular cylindrical container. The vertical-domain-size dependence on the other hand does not yield a definite result when Ra_0 is considered. Hence use of linear stability methods are made and a useful relation arises. This is reflected in Table 1.

The formulation of the problem for the case of thermal convection in bilayers of fluids is considerably more involved. Two important parameters arise – namely the Rayleigh numbers for the lower and upper fluids. We examine the variation of Ra^a, (the Rayleigh number for the lower fluid) while Ra^b is kept fixed, with respect to domain size, boundary conditions and fluid properties. Definite results are obtained for variation with respect to domain size only when a linear stability analysis is also performed. The monotonic dependence of Ra^a on Nusselt number and conductivity ratio (β) is shown in Table 1. It is quite possible to examine the dependence of Ra^b on various parameters by utilizing the above methods. An implicit assumption which is made in the linearised stability analysis is the Rayleigh numbers are such that σ vs β, L_1 or δ curves pass through σ equal to zero. The key tool that helps in the entire analysis is the ability to form a self-adjoint operator with respect to a certain inner product.

References

[1] E.L. Koschmieder, Bénard convection, *Advances in Chemical Physics* (eds. I. Prigogine and S.A. Rice(26, 177–212, Wiley, New York (1974).
[2] R.H. Rogers, Convection, *Rep. Prog. Phys.* 39 (1976), 1–63.
[3] F.H. Busse, Nonlinear properties of convection, *Rep. Prog. Phys.* 41 (1978) 1929–1967.

[4] D.D. Joseph, *Stability of fluid motions,* Vol. II, Springer-Verlag, New York (1976).

[5] D.T.J. Hurle, E. Jakeman and E.R. Pike, On the solution of the Bènard problem with boundaries of finite conductivity, *Proc. Roy. Soc.* A 296 (1967) 469–475.

[6] D.A. Nield, The Rayleigh-Jeffreys problem with boundary slab of finite conductivity, *J. Fluid Mech.* 32 (1968) 393–398.

[7] F.H. Busse and N. Riahi, Nonlinear convection in a layer with nearly insulating boundaries, *J. Fluid Mech.* 96 (1980) 243–256.

[8] K.A. Smith, On convective instability induced by surface tension gradients, *J. Fluid Mech.* 24 (1966) 401–414.

[9] R.W. Zeren and W.C. Reynolds, Thermal instabilities in two fluid horizontal layers, *J. Fluid Mech.* 53 (1972) 305–327.

[10] R. Narayanan, Some differential inequalities for thermohaline convection in bounded containers of arbitrary shape (1982). Submitted for publication.

[11] I. Stakgold, *Green's functions and boundary value problems,* Wiley, New York (1979).

[12] J.M. Mihaljan, A rigorous exposition of the Boussinesq approximation applicable to a thin layer of fluid, *Astrophys. J.* 136 (1982) 1126–1133.

[13] W.V.R. Malkus and G. Veronis, Finite amplitude cellular convection, *J. Fluid Mech.* 4 (1958) 225–260.

[14] A. Schluter, D. Lortz and F. Busse, On the stability of steady finite amplitude celluluar convection, *J. Fluid Mech.* 23 (1965) 129–144.

[15] M. Sherman and S. Ostrach, On the principle of exchange of stabilities for the magnetohydrodynamic thermal stability problem in completely confined fluids, *J. Fluid Mech.* 24 (1966) 661–671.

[16] J.S. Vrentas, R. Narayanan and S.S. Agrawal, Free surface convection in a bounded cylindrical geometry, *Int. J. Heat Mass Transfer* 24 (1981) 1513–1529.

[17] C.Q. Hoard, C.R. Robertson and A. Acrivos, Experiments on the cellular structure in Benard convection, *Int. J. Heat Mass Transfer* 13 (1970) 849–856.

[18] H.J. Palmer and J.C. Berg, Convective instability in liquid pools heated from below, *J. Fluid Mech.* 47 (1971) 779–787.